146 KSME Journal, Vol. 10, No. 2, pp. 146~157, 1996

Vibrational Characteristics of Annular Plates and Rings of Radially Varying Thinkness

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(Received March 9, 1995)

In this paper, annular plates having thickness variation are studied by deriving the equations of motion on the basis of the Mindlin plate theory. The Chebyshev collocation method is employed to solve the differential equation governing the transverse motion of such plates. The dimensionless frequencies are evaluated for different values of taper constant (α), thickness ratio (h_0), radii ratio (ε) and power (n). The results of an experimental investigation are also presented, and the agreement between these findings and the predicted values in theory is remarkably good. As a result of this study, it is found that the effects of rotatory inertia and transverse shear deformation reduce the natural frequencies for all boundary conditions and for all values of n, h_0, ε, α and s (mode number). This study also showed that the natural frequencies of annular plates with thickness expressed by the nth power function are higher than those by the (n-1)th power function for positive values of α, and vice versa for negative values of α for all three boundary conditions. Moreover, there is a proof that the natural frequencies of annular plates tend to be higher as the taper constant decrease and/or as the radii ratio increase for all three boundary conditions and for all values of n, s and h_0.

Key Words: Forced Vibration, Rotor, Natural Frequency, Annular Plate, Inertia, Shear Deformation, Boundary Condition

Nomenclature

- D : Flexural rigidity of plate ($=Eh^3/12(1-\nu^2)$)
- E : Young's modulus of annular plate
- G : Shear modulus of annular plate
- H : Dimensionless variable ($=h/a$)
- K_i : Unknown constants
- M_r, M_t : Bending moments per unit length
- $M_{r\theta}$: Twisting moment per unit length
- Q_r, Q_θ : Radial and tangential shearing forces per unit length
- T : Chebyshev polynomials
- T° : Chebyshev polynomials with superscript meaning integration with respect y
- a : Outer radius
- b : Inner radius
- h : Thickness of plate defined by Eq. (5)
- h_0 : Thickness ratio
- m : Number of Chebyshev collocation point
- n : Power
- q : External force per unit area
- s : Mode number
- t : Time
- x : Dimensionless variable ($=r/a$)
- y : Chebyshev constant defined by Eq. (7b)

Greek Letters

- α : Taper constant
- ε : Ratio of the inner and outer radii (b/a)
- k : Averaging shear coefficient ($=\pi^2/12$)
- ν : Poisson's ratio
- ρ : Density (mass per unit volume)
- ω : Circular frequency
- Ω_s : Dimensionless frequency parameter including the effects of rotatory inertia

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and shear deformation

Ω_c : Dimensionless frequency parameter from classical plate theory

1. Introduction

In recent years, the annular plates of variable thickness have been extensively used in the dynamic design of various machines and structures to reduce their weights and sizes. Considerable work has been done on vibrations of circular plates of the uniform thickness. Leissa (1969) reviewed the work done on vibration of plates. Soni and Rao (1975) studied the vibration of a circular plate with linear taper. Lenox and Conway (1980) obtained an exact, closed form solution for the flexural vibration of a thin annular plate having a parabolic thickness variation. Gupta and Lal (1982) studied free axisymmetric vibrations of polar orthotropic annular plates of a variable thickness. Gorman (1983) applied an annular finite element method to annular discs with variable thickness which have polar orthotropic characteristics. Ramaiah and Vijayakumar (1985) applied the Rayleigh-Ritz method to obtain the natural frequencies of polar orthotropic annular plate. In addition, Kang (1992) treated a Fourier series method for polygonal domains ; large element computation for plates. Lee and Sin (1994) studied the Mindlin plate finite elements by using a modified transverse displacement.

In this paper, the equations of motion of annular plates whose thickness varies with $H=h_0(1-\alpha x^n)$(for $n=1$, 2, 3, and 4) are established with the effects of rotatory inertia and shear deformation included. Since the plate used in the actual practice may have a large thickness, it is important to include the effects of rotatory inertia and shear deformation in order to predict their dynamic behavior with a fair amount of accuracy.

Frequency equations are determined using the Chebyshev collocation method for three combinations of boundary conditions (i.e., C-C, C-S, C-F ; C : clamped edge, S : simply supported edge, F : free edge). And, the frequencies are computed for different values of taper constant (α), radii ratio(ε), thickness ratio(h_0), and power(n) and for the first three modes of vibration in all three cases. The natural frequencies of the annular plates are also investigated by experiment and the theoretical values are compared with experimental results.

2. Theoretical Analysis

The isotropic, homogeneous and elastic annular plates of radially varying thickness are depicted in Fig. 1.

The neutral surface of the annular plate is set on the cylindrical coordinate system(r, θ and z). Consider a plate element dr dθ subject to an external force per unit area q. Then the stress resultant can be exhibited as shown in Fig. 2.

The use of Newton's 2nd law yields the equilib-

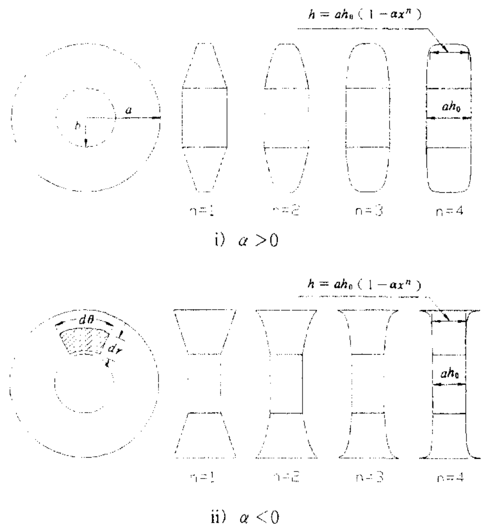


Fig. 1 Annular plates of radially varing thickness.

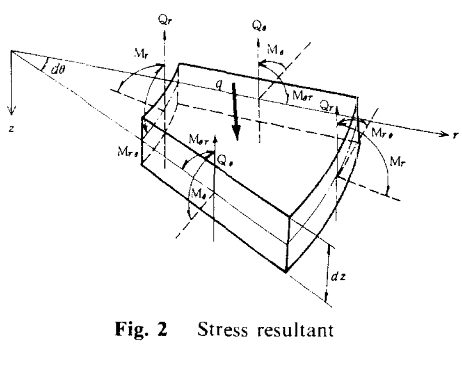


Fig. 2 Stress resultant

rium equations.

These equations can be simplified as follows:

$$\frac{\partial M_r}{\partial r}+\frac{M_r-M_\theta}{r}+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta}-Q_r$$
$$=\frac{\rho h^3}{12}\frac{\partial^2 \phi_r}{\partial t^2} \quad (1a)$$

$$\frac{\partial M_{r\theta}}{\partial r}+\frac{2M_{r\theta}}{r}+\frac{1}{r}\frac{\partial M_\theta}{\partial \theta}-Q_\theta$$
$$=\frac{\rho h^3}{12}\frac{\partial^2 \phi_\theta}{\partial t^2} \quad (1b)$$

$$\frac{\partial Q_r}{\partial r}+\frac{1}{r}\frac{\partial Q_\theta}{\partial \theta}+\frac{Q_r}{r}+q=\rho h\frac{\partial^2 w}{\partial t^2} \quad (1c)$$

The right hand side terms of Eqs. (1a) and (1b) are the rotatory inertia of the element. In the classical plate theory these terms are neglected.

Equations (1a), (1b) and (1c) are the equations of motion of an annular plate when an external force q is applied (Here, the equations of motion for free vibration are obtained when $q=0$).

For the axisymmetric motion, Eqs. (1a), (1b) and (1c) become

$$\frac{\partial M_r}{\partial r}+\frac{M_r-M_\theta}{r}-Q_r=\frac{\rho h^3}{12}\frac{\partial^2 \phi_r}{\partial t^2} \quad (2a)$$

$$\frac{Q_r}{r}+\frac{\partial Q_r}{\partial r}=\rho h\frac{\partial^2 w}{\partial t^2} \quad (2b)$$

where

$$M_r=D\left(\frac{\partial \phi_r}{\partial r}+\frac{\nu}{r}\phi_r\right)$$
$$M_\theta=D\left(\frac{\partial \phi_r}{\partial r}+\nu\frac{\partial \phi_r}{\partial r}\right)$$
$$Q_r=kGh\left(\phi_r+\frac{\partial w}{\partial t}\right) \quad (2c)$$

For harmonic motion,

$$w(r,t)=\bar{w}(r)e^{iwt} \text{ and}$$
$$\phi_r(r,t)=\phi(r)e^{iwt} \quad (3)$$

Having substituted Eqs. (2c) and (3) into Eqs. (2a) and (2b) and rearranging, the results can be rewritten as follows:

$$\Psi-12kGh\frac{(1-\nu^2)}{E}\left(\phi+\frac{d\bar{w}}{dr}\right)=0 \quad (4a)$$

$$\frac{d}{dr}[r\Psi]+12\rho\, rh\frac{(1-\nu^2)}{E}\,\omega^2\bar{w}=0 \quad (4b)$$

where

$$\Psi=h^3\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}-\frac{\phi}{r^2}\right)+3h^2\frac{dh}{dr}\left(\frac{d\phi}{dr}+\frac{\nu}{r}\phi\right)+\rho h^3\frac{(1-\nu^2)\omega^2}{E}\,\phi$$

Consider an annular plate whose thickness is expressed as power functions. That is,

$$h=ah_0(1-\alpha x^n)(n=1,\ 2,\ 3,\ \text{and } 4) \quad (5)$$

When the numerical value of n is very large the thickness suddenly varies near the edge. Since such plates are not common in practice, the values $n=1$, 2, 3, and 4 are considered in this paper. The plate thickness changes linearly for $n=1$ and changes parabolically for $=2$.

Elimination of $\bar{w}$ from Eq. (4) and substitution of Eq. (5) into the new equation leads to an uncoupled differential equation in ϕ. The resulting equation for the thickness variation in the radial direction is reduced to the non-dimensional 4th order linear homogeneous differential equation with variable coefficients as follows:

$$A_0\frac{d^4\phi}{dx^4}+A_1\frac{d^3\phi}{dx^3}+A_2\frac{d^2\phi}{dx^2}+A_3\frac{d\phi}{dx}+A_4\phi=0 \quad (6)$$

Case 1) When $n=1$, that is, when $H=h_0(1-\alpha x)$ the coefficients A_0 to A_4 can be determined as follows:

$$A_0=(1-\alpha x)^2$$
$$A_1=\frac{2}{x}(1-\alpha x)(1-5\alpha x)$$
$$A_2=\frac{1}{x^2}[(1-\alpha x)^2\{\Omega_S^2x^2(1-\frac{1}{k_0})-3\}-\alpha x(1-\alpha x)(10+3\nu)+12\alpha^2x^2]$$
$$A_3=\frac{1}{x^3}[(1-\alpha x)^2\{\Omega_S^2x^2(1+\frac{1}{k_0})+3\}+\alpha x(1-\alpha x)\{6+3\nu+5-\Omega_S^2x^2(5+\frac{3}{k_0})\}+3\alpha^2x^2(3\nu+2)]$$
$$A_4=\frac{1}{x^4}[(1-\alpha x)^2\{\Omega_S^2x^2(\Omega_S^2x^2-1)\frac{1}{k_0}-1)-3\}-\alpha x(1-\alpha x)(8+\Omega_S^2x^2(2+3\frac{\nu}{k_0})-3\alpha^2x^2(1+2\nu-\Omega_S^2x^2)-\frac{12}{h_0^2}\Omega_S^2x^4] \quad (6a)$$

where

$$\Omega_S^2=\frac{\rho a^2\omega^2(1-\nu^2)}{E},\ x=\frac{r}{a},\ H=\frac{h}{a}$$
$$k_0=\frac{kG(1-\nu^2)}{E},\ W=\frac{w}{a}$$

Case 2) When $n=2$, that is, when $H=h_0(1-\alpha x^2)$ the coefficients A_0 to A_4 can be determined as follows:

$$A_0=(1-\alpha x^2)^2$$
$$A_1=\frac{2}{x}(1-\alpha x^2)(1-9\alpha x^2)$$
$$A_2=\frac{1}{x^2}[(1-\alpha x^2)^2\{\Omega_S^2x^2(1+\frac{1}{k_0})-3\}-(1-\alpha x^2)\{2\alpha x^2(10+3\nu)+18\alpha x^2\}+48\alpha^2x^4]$$
$$A_3=\frac{1}{x^3}[(1-\alpha x^2)^2\{\Omega_S^2x^2(1+\frac{1}{k_0}+3)\}+(1-\alpha x^2)\{2\alpha x^2(11+3\nu-\Omega_S^2x^2(5+3\frac{1}{k_0}))+12\alpha x^2(1+\nu)\}+12\alpha^2x^4(3\nu+2)+60\alpha^2x^4]$$
$$A_4=\frac{1}{x^4}[(1-\alpha x^2)^2\{\Omega_S^2x^2((\Omega_S^2x^2-1)\frac{1}{k_0}-1)-3\}-(1-\alpha x^2)\{2\alpha x^2(8+\Omega_S^2x^2(2+3\frac{\nu}{k_0}))-6\alpha x^2(1+\nu)\}-12\alpha^2x^4(1+2\nu-\Omega_S^2x^2)-\frac{12}{h_0^2}\Omega_S^2x^4+60\alpha^2x^4\nu] \tag{6b}$$

Case 3) When $n=3$, that is, when $H=h_0(1-\alpha x^3)$ the coefficients A_0 to A_4 can be determined as follows:

$$A_0=(1-\alpha x^3)^2$$
$$A_1=\frac{2}{x}(1-\alpha x^3)(1-13\alpha x^3)$$
$$A_2=\frac{1}{x^2}[(1-\alpha x^3)^2\{\Omega_S^2x^2(1+\frac{1}{k_0})-3\}+(1-\alpha x^3)\{-3\alpha x^3(10+3\nu)-54\alpha x^3\}+108\alpha^2x^6]$$
$$A_3=\frac{1}{x^3}[(1-\alpha x^3)^2\{\Omega_S^2x^2(1+\frac{1}{k_0}+3)\}-(1-\alpha x^3)\{-3\alpha x^3(11+3\nu)-\Omega_S^2x^2(5+\frac{3}{k_0})\}-36\alpha x^3(1+\nu)-18\alpha x^3\}+27\alpha^2x^6(3\nu+2)+270\alpha^2x^6]$$
$$A_4=\frac{1}{x^4}[(1-\alpha x^3)^2\{\Omega_S^2x^2((\Omega_S^2x^2-1)\frac{1}{k_0}-1)-3\}+(1-\alpha x^3)\{-3\alpha x^3(8+\Omega_S^2x^2(2+3\frac{\nu}{k_0}))+18\alpha x^3(1+\nu)-18\alpha x^3\nu\}-27\alpha^2x^6(1+2\nu-\Omega_S^2x^2)-\frac{12}{h_0^2}\Omega_S^2x^4+270\alpha^2x^6\nu] \tag{6c}$$

Case 4) When $n=4$, that is, when $H=h_0(1-\alpha x^4)$ the coefficients A_0 to A_4 can be determined as follows:

$$A_0=(1-\alpha x^4)^2$$
$$A_1=\frac{2}{x}(1-\alpha x^4)(1-17\alpha x^4)$$
$$A_2=\frac{1}{x^2}[(1-\alpha x^4)^2\{\Omega_S^2x^2(1+\frac{1}{k_0})-3\}+(1-\alpha x^4)\{-4\alpha x^4(10+3\nu)-108\alpha x^4\}+192\alpha^2x^8]$$
$$A_3=\frac{1}{x^3}[(1-\alpha x^4)^2\{\Omega_S^2x^2(1+\frac{1}{k_0})+3\}-(1-\alpha x^4)\{-4\alpha x^4(11+3\nu)-\Omega_S^2x^2(5+\frac{3}{k_0}))-72\alpha x^4(1+\nu)-72\alpha x^4\}+48\alpha^2x^8(3\nu+2)+720\alpha^2x^8]$$
$$A_4=\frac{1}{x^4}[(1-\alpha x^4)^2\{\Omega_S^2x^2((\Omega_S^2x^2-1)\frac{1}{k_0}-1)-3\}+(1-\alpha x^4)\{-4\alpha x^4(8+\Omega_S^2x^2(2+\frac{3\nu}{k_0}))+36\alpha x^4(1+\nu)-72\alpha x^4\nu\}-48\alpha^2x^8(1+2\nu-\Omega_S^2x^2)-\frac{12\Omega_S^2}{h_0^2}x^4+720\alpha^2x^8\nu] \tag{6d}$$

3. Numerical Analysis

3.1 Method of solution

Equation (6) is a linear differential equation with variable coefficients. Its solution can not be easily found in closed form. The Chebyshev collocation method has been used in this paper. Since this method is applicable only in the interval $(-1, 1)$, the range of the differential Eq. (6) can be transformed from the interval $(\varepsilon, 1)$ to $(-1, 1)$ by the following transformation

$$x\triangleq\frac{(1-\varepsilon)y+(1+\varepsilon)}{2} \tag{7a}$$

Where, y is defined as

$$y=y_j=\cos\left(\frac{2j+1}{m-4}\cdot\frac{\pi}{2}\right)\quad (j=1, 2, 3, \cdots, m-4) \tag{7b}$$

Substitution of Eq. (7a) into Eq. (6) yields

$$F_0\frac{d^4\phi}{dy^4}+F_1\frac{d^3\phi}{dy^3}+F_2\frac{d^2\phi}{dy^2}+F_3\frac{d\phi}{dy}+F_4\phi=0$$
$$F_0=A_0\left(\frac{2}{1-\varepsilon}\right)^4,\ F_1=A_1\left(\frac{2}{1-\varepsilon}\right)^3$$
$$F_2=A_2\left(\frac{2}{1-\varepsilon}\right)^3,\ F_3=A_3\left(\frac{2}{1-\varepsilon}\right)$$

$$F_4 = A_4 \tag{8}$$

where, A_0 to A_4 are variable coefficients of Eq. (6).

According to the Chebyshev collocation method,

$$\frac{d^4\phi}{dy^4} = \sum_{i=5}^{m} K_i T_{i-5} \tag{9a}$$

where, K_i's(i=5, 6, ⋯, m) are unknown constants, and T_i's are Chebyshev polynomials defined as

$$T_0 = 1,\ T_1 = y,\ T_2 = 2y^2 - 1,\ \cdots\cdots,$$
$$T_i = 2yT_{i-1} - T_{i-2},\ i \geqq 2$$

If Eq. (9a) is iteratively integrated with respect to y,
ϕ and its derivatives can be expressed in terms of T_i and K_i.

$$\frac{d^3\phi}{dy^3} = K_4 + \sum_{i=5}^{m} K_i T_{i-5}^{(1)} \tag{9b}$$

$$\frac{d^2\phi}{dy^2} = K_3 + K_4 T_1 + \sum_{i=5}^{m} K_i T_{i-5}^{(2)} \tag{9c}$$

$$\frac{d\phi}{dy} = K_2 + K_3 T_1 + K_4 T_1^{(1)} + \sum_{i=5}^{m} K_i T_{i-5}^{(3)} \tag{9d}$$

$$\phi = K_1 + K_2 T_1 + K_3 T_1^{(1)} + K_4 T_1^{(2)} + \sum_{i=5}^{m} K_i T_{i-5}^{(4)} \tag{9e}$$

where, K_1, K_2, K_3 and K_4 are the constants of integration.

The superscript over T denoting the integration with respect to y is defined as follows :

$$T_i^{(1)} = \int T_i dy = \frac{1}{2}\left[\frac{T_{i+1}}{i+1} - \frac{T_{i-1}}{i-1}\right],\ i > 1,$$

$$T_0^1 = T_1,\ T_1^1 = \frac{T_2 + T_0}{4}$$

Having substituted Eqs. (9a~e) into Eq. (6), the final equation for m (number of Chebyshev collocation point) can be exhibited in a matrix form.

$$[A_{l1}\ A_{l2}\ A_{l3} \cdots A_{lm}] \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (l = 1, 2, 3, \cdots, m-4)$$

or in a simple form,

$$[N]\{K\} = \{0\} \tag{10}$$

Where, $[N]$ is an (m−4) by m matrix, and $\{K\}$ is a column vector of order m. Other four equations can be obtained by employing the boundary conditions at the inner and outer edges of annulus ; $y=-1$ and $y=1$ respectively. Thus, m equations and m unknown constants are obtained.

3.2 Boundary conditions and characteristic equation

The following combinations of boundary conditions are considered ;

i) both inner and outer edges clamped(C−C)

$$W(y)\,|_{y=\pm 1} = 0$$
$$\phi(y)\,|_{y=\pm 1} = 0$$

These equations can be expressed in a matrix form as

$$[N_{cc}]\{K\} = \{0\} \tag{11}$$

where, $[N_{cc}]$ is a 4 by m matrix, and $\{K\}$ is a column vector of order m.

ii) inner edge clamped, outer edge simply supported(C−S)

$$W(y)\,|_{y=\pm 1} = 0,$$
$$\left(\frac{2}{1-\varepsilon}\right)\frac{d\phi}{dy} + \frac{\nu}{x}\phi\,|_{y=1} = 0$$
$$\phi(y)\,|_{y=-1} = 0$$

or in a matrix form

$$[N_{cs}]\{K\} = \{0\} \tag{12}$$

where, $[N_{cs}]$ is a 4 by m matrix, and $\{K\}$ is a column vector of order m.

iii) inner edge clamped, outer edge free(C−F)

$$W(y)\,|_{y=-1} = 0,$$
$$\left(\frac{2}{1-\varepsilon}\right)\frac{d\phi}{dy} + \frac{\nu}{x}\phi\,|_{y=1} = 0$$
$$\phi(y)\,|_{y=-1} = 0,\quad \phi + \left(\frac{2}{1-\varepsilon}\right)\frac{dW}{dy}\,|_{y=1} = 0$$

or in a matrix form

$$[N_{cf}]\{K\} = \{0\} \tag{13}$$

where, $[N_{cf}]$ is a 4 by m matrix, and $\{K\}$ is a column vector of order m.

Combining Eqs. (11), (12), (13) and (10), we obtain the following equations.

$$\text{C-C case : } \begin{bmatrix} N \\ N_{CC} \end{bmatrix}_{m\times m} \{K\}_{m\times 1} = \{0\} \tag{14}$$

$$\text{C-S case : } \begin{bmatrix} N \\ N_{CS} \end{bmatrix}_{m\times m} \{K\}_{m\times 1} = \{0\} \tag{15}$$

C-F case : $$\left[\frac{N}{N_{CF}}\right]_{m\times m}\{K\}_{m\times 1}=\{0\} \qquad (16)$$

For nontrivial solutions of Eqs. (14), (15) and (16), the frequency determinant must be vanished and hence we obtain

$$\left|\frac{N}{N_{CC}}\right|=0 \qquad (17)$$

$$\left|\frac{N}{N_{CS}}\right|=0 \qquad (18)$$

$$\left|\frac{N}{N_{CF}}\right|=0 \qquad (19)$$

Equations (17), (18) and (19) are thus characteristic equations for the C-C, C-S and C-F cases, respectively.

3.3 Numerical evaluation and discussion

Since a plate is an example of a continuous system, infinitely many roots of a frequency

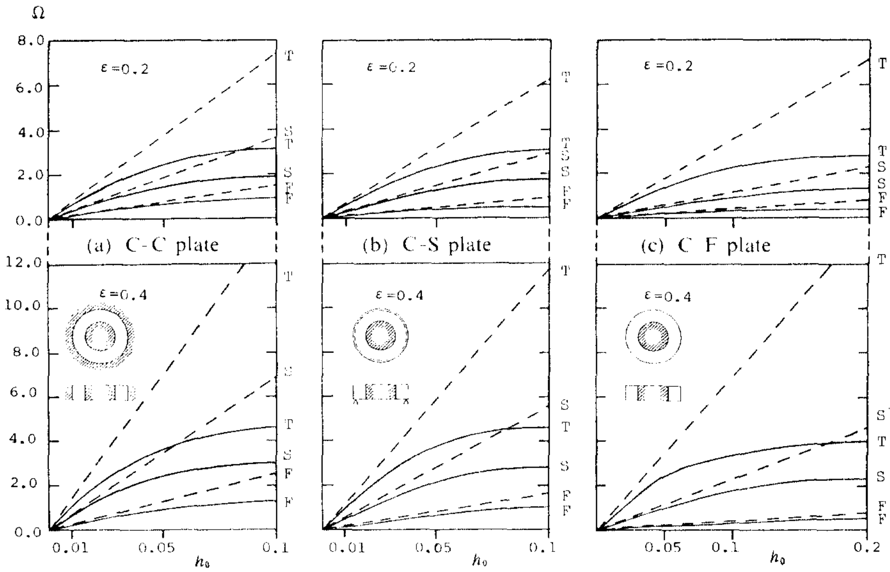


Fig. 3 Dimensionless frequency parameter vs. thickness ratio for n=1 and α=−0.5. ————, shear theory ; ············, classical theory ; F, 1st mode ; S, 2nd mode ; T, 3rd mode.

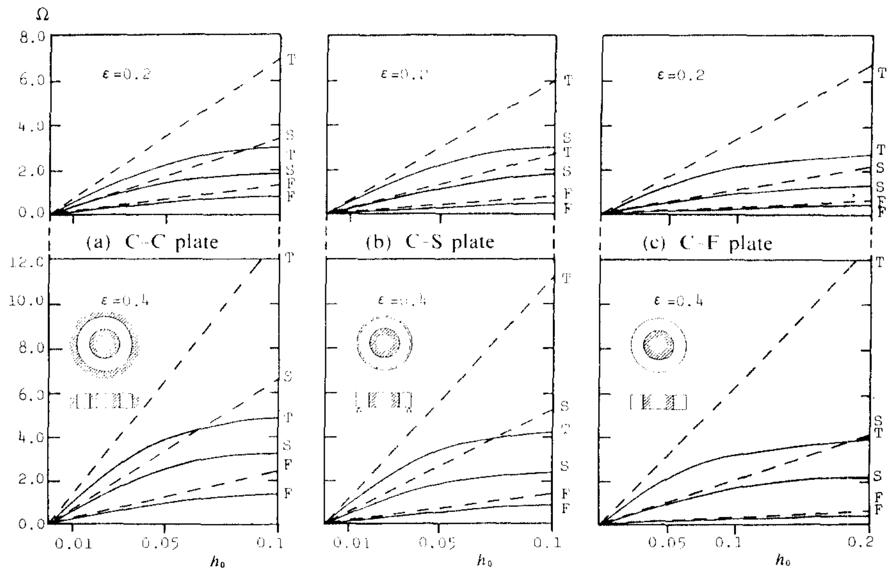


Fig. 4 Dimensionless frequency parameter vs. thickness ratio for n=2 and α=−0.5. ————, shear theory ; ············, classical theory ; F, 1st mode ; S, 2nd mode ; T, 3rd mode.

parameter can be obtained. The frequency parameters are computed for the first three modes of vibration according to the present theory. The number of collocation points (i. e., m) is fixed at 13, since further increase in m does not noticeably improve the results.

Frequency parameters Ω_S and Ω_C are computed for all three cases of the boundary condition ;

(i) when the radii ratio(ε) varies form 0.1 to 0.7 by an increment of 0.1,

(ii) when the thickness ratio(h_0) varies form 0.01 to 0.2 by an increment of 0.01,

(iii) when the taper constant(α) varies form -0.7 to 0.7 by an increment of 0.2(for $n=1$, 2, 3 and

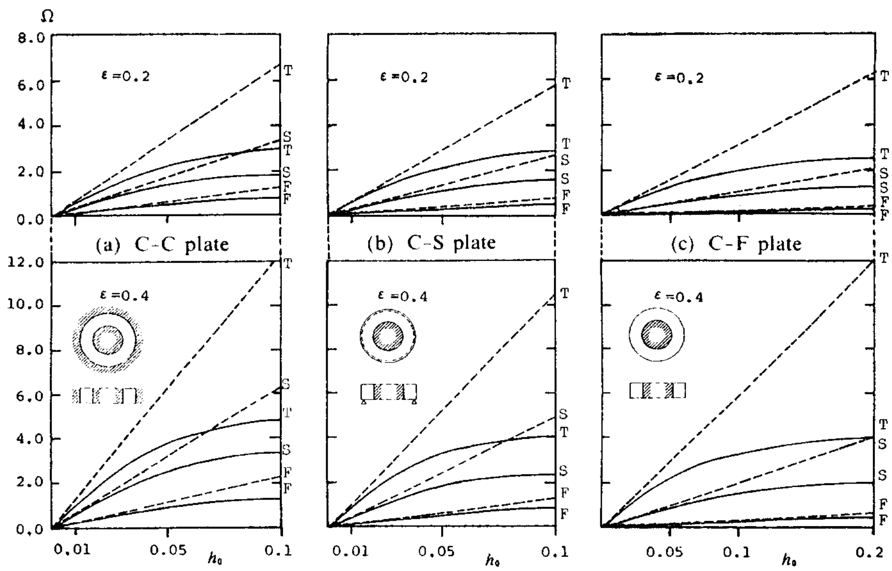


Fig. 5 Dimensionless frequency parameter vs. thickness ratio for n=3 and $\alpha=-0.5$. ————, shear theory ; ············, classical theory ; F, 1st mode ; S, 2nd mode ; T, 3rd mode.

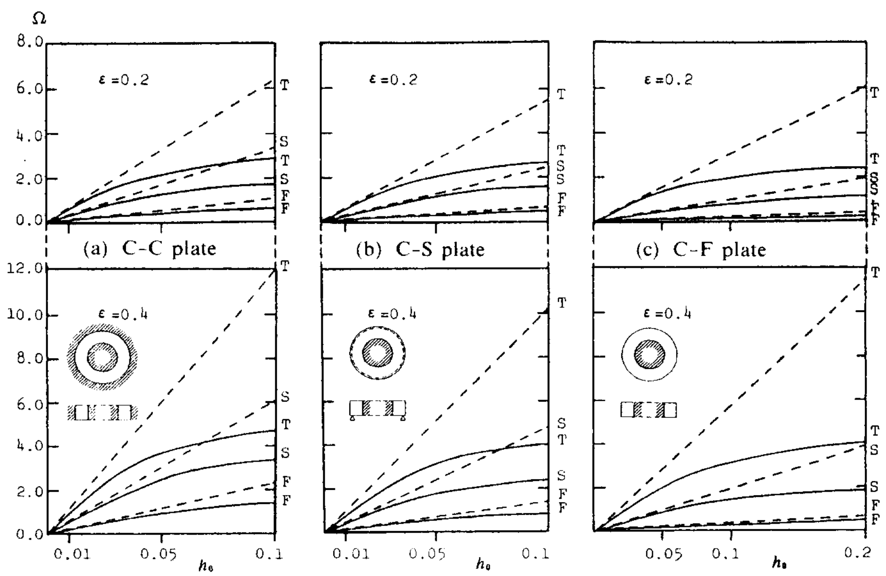
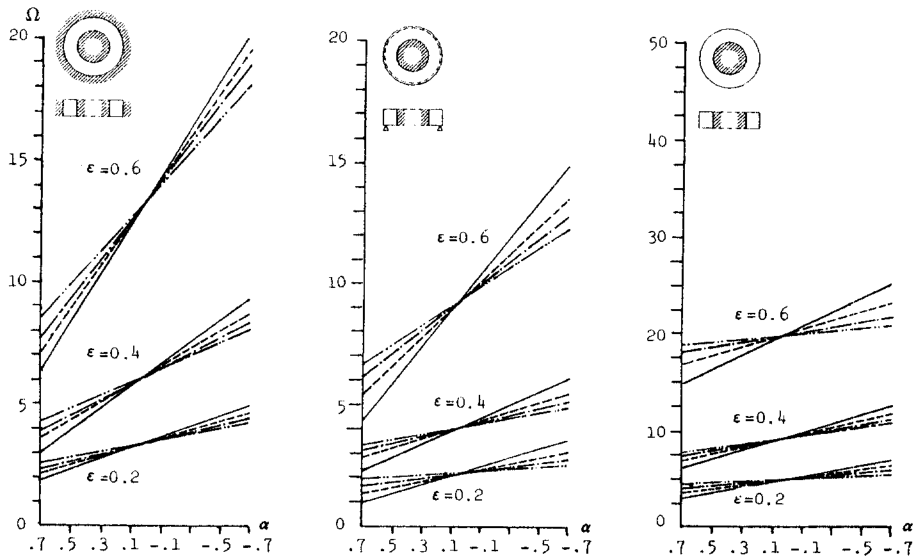


Fig. 6 Dimensionless frequency parameter vs. thickness ratio for n=4 and $\alpha=-0.5$. ————, shear theory ; ············, classical theory ; F, 1st mode ; S, 2nd mode ; T, 3rd mode.

4).

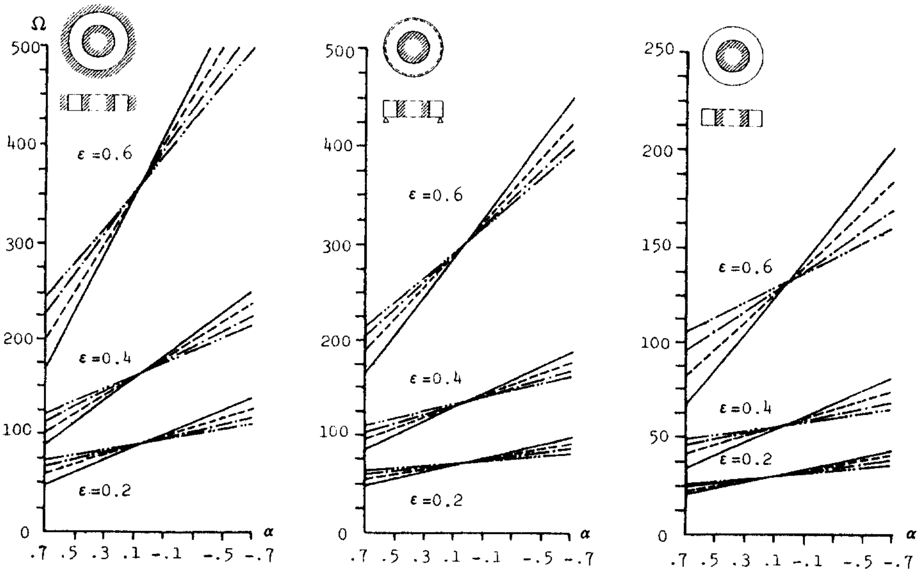
Figures 3, 4, 5 and 6 represent the variation of the frequency parameter with the thickness ratio(h_0) for $\alpha=-0.5$, for C-C, C-S and C-F plates, respectively. From these figures, it is observed that the difference between Ω_S and Ω_C turns out to be increased for all values of power(n) in the following manner:

(i) with increase of thickness ratio(h_0),
(ii) with increase of mode number(s),
(iii) with decrease of taper constant(α),
(iv) with increase of radii ratio(ε).



(a) C-C plate (b) C-S plate (c) C-F plate

Fig. 7 Dimensionless frequency parameter vs. taper constant for the 1st mode. ———, n=1; ·········, n=2; —·—·, n=3; —··—··, n=4.



(a) C-C plate (b) C-S plate (c) C-F plate

Fig. 8 Dimensionless frequency parameter vs. taper constant for the 2nd mode ———, n=1; ·········, n=2; —·—·, n=3; —··—··, n=4.

Figures 7 and 8 prove that the natural frequencies of annular plates with thickness expressed by the nth power function are higher than those by the $(n-1)$th power function for positive values of α, and vice versa for negative values of α for all three boundary conditions and for all values of s, ε and h_0. Also the frequency parameters(Ω_S, Ω_C) for a C-S plate are larger than those for C-F plate but are smaller than those for a C-C plate for all values of n, s, α, ε and h_0.

Furthermore, Figs. 7, 8 and 9 show that the natural frequencies of annular plates become higher as the radii ratio(ε) increases and/or as the taper constant(α) decreases for all three

Table 1 Comparison between the results of present theory and those of Vogel and Skinner for annular plate of $\alpha=0$, dimension in (rad/s)

	s \ ε	0.1	0.3	0.5	0.7
S.M. VOGEL	1	5278.1	8765.	17274.7	47985.3
Present Theory	1	5210.6	8776.5	17280.9	48098.3
S.M. VOGEL	2	14587.5	24280.	47665.4	132759.4
Present Theory	2	15044.4	24168.2	47451.9	131910.8

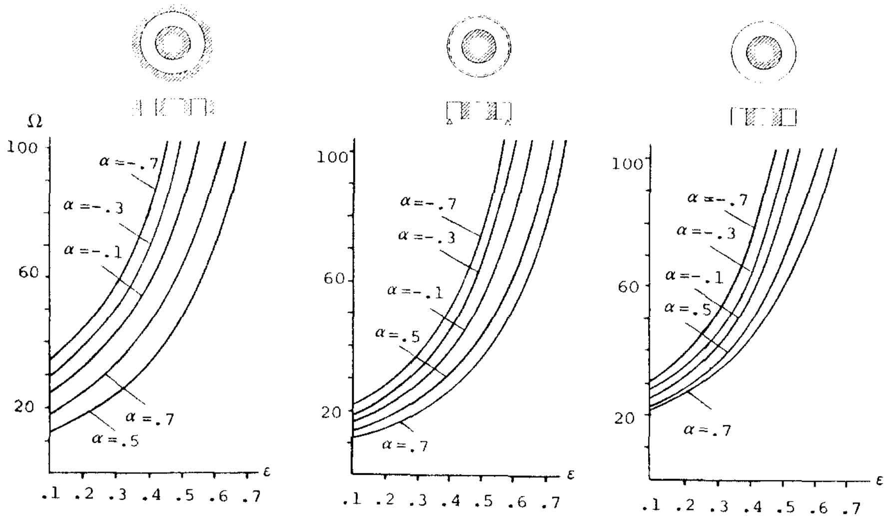
(a) Clamped — Clamped

	s \ ε	0.1	0.3	0.5	0.7
S.M. VOGEL	1	3454.9	5790.2	11580.2	32630.8
Present Theory	1	3437.3	5803.3	11581.9	32628.7
S.M. VOGEL	2	11644.9	19450.4	38388.3	106847.3
Present Theory	2	11713.4	19404.7	38273.5	106787.9

(b) Clamped — Simply Supported

	s \ ε	0.1	0.3	0.5	0.7
S.M. VOGEL	1	818.9	1289.2	2520.8	7165.8
Present Theory	1	821.1	1291.4	2521.6	7154.5
S.M. VOGEL	2	4894.5	8253.5	16474.9	46385.8
Present Theory	2	5026.5	8236.2	16465.7	46435.2

(c) Clamped — Free



(a) C-C plate (b) C-S plate (c) C-F plate

Fig. 9 Dimensionless frequency parameter vs. radii ratio for n=2

boundary conditions and for all values of n, s and h_0.

Finally, in Table 1 a comparison of the results of present theory with those of S. M. Vogel and D. W. Skinner (they used Bessel function for the numerical analysis of annular plate with uniform thickness (α=0)) is summarized in order to verify the accuracy of the solution presented in this paper. This table shows that the Chebyshev collocation method is very reliable because the difference is within 1.0% ranging over 0.2~1.0%.

4. Experiment

4.1 Experimental procedure and apparatus

In this paper, the natural frequencies of the annular plates are detected by the measuring system shown in Fig. 10 to confirm the validity of the theoretical values.

By impacting the annular plates with an impulse hammer, their signals are input to F.F.T. analyzer through amplifier and then spectrums of transfer functions are displayed through data display unit.

The material used in this experiment is SM45C. Its Young's modulus(E), density(ρ), and Poisson's ratio(ν) are equal to $2.1\times10^{10}(kg_f/m^2)$, $810(kg_f\ s/m^4)$ and 0.3, respectively. The inner and outer radii of the annular plates are 0.045m and 0.15m, respectively. The experiments are conducted for C-F plate, for α=0, −0.5, −0.7, n=2 and ε=0.3.

4.2 Experimental results and analysis

Figure 11 shows the frequency spectrums appeared in F.F.T. analyzer. In Table 2, the experimental results are compared with the theoretical values. The results agree within a few percent of the theoretical values. The difference between experimental results and theoretical values may be resulted from an imperfect supporting

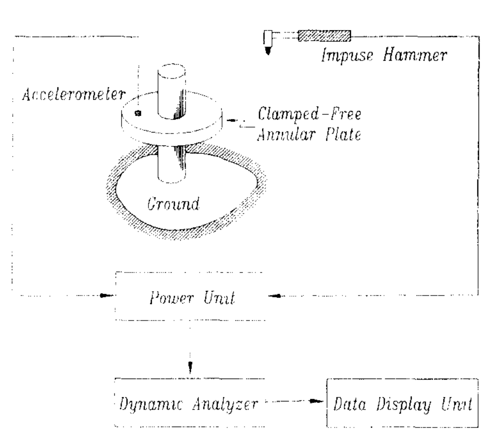


Fig. 10 Schematic diagream of measuring system

Table 2 Comparison between experimental results and the theoretical values for n=2 and ε=0.3

α	s	Experimental Frequency (rad/s) (EF)	Calculated Frequency by Classical Theory (rad/s) (CFCT)	Calculated Frequency by Present Theory (rad/s) (CFPT)	Difference (EF, CFCT) (%)	Difference (EF, CFPT) (%)
0	1	1245.1	1386.5	1283.5	10.2	3.1
	2	8220.3	9880.2	8392.6	16.8	2.1
-0.5	1	1325.7	1445.1	1342.2	8.2	1.3
	2	9150.4	11266.8	9415.3	19.5	2.9
-0.7	1	1350.5	1520.8	1387.8	11.2	2.8
	2	9800.8	12359.1	9889.0	20.7	0.9

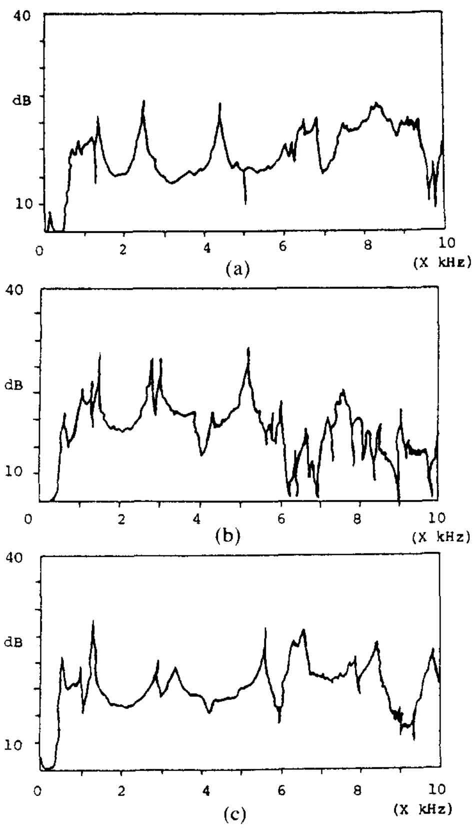


Fig. 11 Frequency spectrums of annular plate for (a) α =0, (b) $\alpha=-0.5$, (c) $\alpha=-0.7$ with Clamped-Free boundary condition.

condition. Furthermore, if the damping effect were included in the present theory, the difference should turn out to be much closer.

7. Conclusions

The dynamic characteristics of annular plates with variable thickness are both theoretically and experimentally analyzed. An analysis and a comparison between theoretical and experimental results are as follows:

① The values of Ω_S (including the effects of rotatory inertia and shear deformation) are always smaller than those of Ω_C (from the classical theory) for all boundary conditions and for all values of radii ratio(ε), thickness ratio(h_0), tapper constant(α), power(n) and mode number(s). Furthermore, the difference between Ω_S and Ω_C turns out to be increased with decreasing in α and increasing in h_0, ε and s.

② The natural frequencies of annular plates are higher when its thickness is expressed by the nth power function than those by the $(n-1)$th power function for positive values of α, and vice versa for negative values of α, for all three boundary conditions and for all values of ε, h_0 and s.

③ The natural frequencies of annular plates tend to increase as taper constant(α) decreases and/or as radii ratio(ε) increases for all boundary conditions and for all values of h_0, n and s.

④ The frequency parameters(Ω_S, Ω_C) for a C-S plate are larger than those for a C-F plate but are smaller than those for a C-C plate for all values of ε, h_0, α, n and s.

Acknowledgment

This work was partly supported by the Korea Science and Engineering Foundation under Grant No. 93-0500-03-03-3. The author would like to thank for the financial support of this work.

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